AN INTRODUCTION TO SINGULAR LIMITS IN PHASE SEPARATION: FROM DIFFUSE TO SHARP INTERFACE

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Phase separation phenomenon can be described as the formation of two distinct phases from a single homogeneous mixture. It is ubiquitous in nature: from polymer mixtures to Cell Biology. The main modeling approaches for this phenomenon aim at considering the interfaces separating the different phases as to be either sharp, or admitting a finite size transition layer. The former approach naturally leads to free boundary problems explicitely tracking interface evolution, whereas the latter gives rise to a PDE problem where the phase state is included implicitly in the governing equations. Although apparently very different, the two approaches are tightly related, and these connections have been attracting the attention of a vast community of researchers for more than fifty years.

The goal of this course is to introduce some of the most interesting relations between the diffuse and sharp interface models. We will start from a recap on the basic notions of Geometric Measure Theory and Bochner-Sobolev spaces, to develop some fundamental tools, which are necessary to move on to consider the Allen-Cahn equation. We will study this equation as an L^2-gradient flow of the Ginzburg-Landau energy functional, showing its well-posedness in the weak solution setting. Since this part is more PDE oriented, if the students are interested, we can also give some insights on how to extend these arguments to deal with hydrodynamic couplings like the Navier-Stokes-Allen-Cahn equation. Then, in order to understand the relation between diffuse and sharp interface, we turn to the celebrated Modica-Mortola Γ-convergence result [3] for vanishing inferface thickness parameter, connecting the Ginzburg-Landau functional to the (finite) perimeter of the separating interface. In conclusion, the last two lectures will be devoted to give the flavor of more advanced results about the sharp interface limit of the complete Allen-Cahn equation to the so-called Mean Curvature Flow (MCF). We will explore the difficulties arising in such a singular limit procedure leading to two different results. Namely, the first one, under a suitable energy convergence assumption, leads to a BV solution to MCF (see, e.g., [2]), whereas a second one, albeit not conditional, only leads to the weaker notion of a varifold solution to MCF (we refer to the classical [1]). Some hot open problems and possible further developments of the theory will also be mentioned during the course.

Tentative schedule:

Lecture 1 Recap on Geometric Measure Theory and Bochner-Sobolev spaces

Lecture 2 Well-posedness of the Allen-Cahn equation (AC)

Lecture 3 Modica-Mortola Γ-convergence result

Lecture 4 Sharp interface limit of AC: BV solutions to Mean Curvature Flow (MCF)

Lecture 5 Sharp interface limit of AC: Varifold solutions to MCF

References

[1] T. Ilmanen, *Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature*, J. Differential Geom., 38 (1993), pp. 417–461.

[2] T. Laux and T. M. Simon, *Convergence of the Allen-Cahn equation to multiphase mean curvature flow*, Comm. Pure Appl. Math., 71 (2018), pp. 1597–1647.

[3] L. Modica and S. Mortola, *Un esempio di Gamma-convergenza*, Bollettino della Unione Matematica Italiana B, 14 (1977), pp. 285–299.